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ON GENERALIZED DISTANCE SECURE SETS IN GRAPHS

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Abstract. Let S be a non empty subset of vertex set. For any $v \in S$, the vertex v and its neighbors inside S are called defenders of v, whereas those lying outside S are attackers of v with respect to S. An attack on S is a collection of mutually disjoint sets of attackers of vertices in S, whereas a defense is that of defenders. An attack on S is defendable if there is a defense of S such that for every vertex in S, the cardinality of corresponding defending set is more than or equal to that of corresponding attacking set. The set S is a secure set if every attack on S is defendable. An ultra secure set is a secure set in which every attack is defendable by a single defense of S. In this paper, various types of generalized distance secure sets are introduced by considering attackers and defenders from distances more than 1. A characterization of k-distance ultra secure sets is obtained and some properties of generalized distance secure sets are discussed.

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1. Introduction

In a simple connected graph $G = (V, E)$, let $S = \{s_1, s_2, \ldots, s_m\} \subseteq V$. For any $s_i \in S$, the sets $N[s_i] - S$ and $N[s_i] \cap S$ are called the set of attackers and the set of defenders of s_i respectively. A collection $\mathcal{A} = \{A_i : A_i \subseteq N[s_i] - S, 1 \leq i \leq m\}$ is an attack[1] on S if $A_i \cap A_j = \emptyset$ for all $i \neq j$, $1 \leq i, j \leq m$. A defense[1] of S is a collection $\mathcal{D} = \{D_i : D_i \subseteq N[s_i] \cap S, 1 \leq i \leq m\}$, where $D_i \cap D_j = \emptyset$ for all $i \neq j$, $1 \leq i, j \leq m$. For a set $S = \{s_i : 1 \leq i \leq m\} \subseteq V$, an attack $\mathcal{A} = \{A_i : 1 \leq i \leq m\}$ on S is said to be defendable if there is a defense $\mathcal{D} = \{D_i : 1 \leq i \leq m\}$ of S such that $|D_i| - |A_i| \geq 0 \ \forall\, i$, $1 \leq i \leq m$. The set S is a secure set[1] if every attack on S is defendable. More results on secure sets can be found in [1, 3, 4, 7].

A new approach of secure sets is given in [8, 9]. For any $S \subseteq V$, $Bord(S) = \{v \in S : N[v] \cap (V - S) \neq \emptyset\}$ is the border of S. An attack $\mathcal{A}$ on S is maximal if $\bigcup\limits_{A \in \mathcal{A}} A = N[S] - S$. A defense $\mathcal{D}$ of S is maximal if $\bigcup\limits_{D \in \mathcal{D}} D = N[Bord(S)] \cap S$. In [9], it has been shown that there exists a one to one correspondence between the maximal attacks on S and the functions $A : N[S] - S \to Bord(S)$ satisfying the condition that for any $x \in N[S] - S$, x and $A(x)$ are adjacent. Similar correspondence exists between the maximal defenses of S

and the functions $D : N[Bord(S)] \cap S \to Bord(S)$ satisfying the condition that for any $y \in N[Bord(S)] \cap S$, y and $D(y)$ are equal or adjacent. Further it has been shown that a set S is a secure set if and only if every maximal attack on S is defendable. In [9], these notions have been generalized and k-distance secure sets are defined by considering attackers and defenders from distance greater than 1.

Let k be an integer with $1 \leq k \leq \text{diam}(G) = max\{d(u,v) : u,v \in V\}$. For any $v \in V$ and non negative integer k, the sets $N_k(v) = \{u \in V : 0 < d(u,v) \leq k\}$ and $N_k[v] = \{u \in V : 0 \leq d(u,v) \leq k\}$ are called the open k-neighborhood and closed k-neighborhood of v respectively. For any $S \subseteq V$, $N_k[S] = \underset{v\in S}{\cup} N_k[v]$. For any subgraph H, the vertex set and the edge set of H are denoted by $V(H), E(H)$ respectively. For any $u, v \in V(H)$, $d_H(u,v)$ denotes the distance between u and v in H. For any $S \subseteq V(H)$ and $v \in V(H)$, the notations $N_k^H[v]$ and $N_k^H[S]$ denote the respective k-neighborhoods in the graph H. We may not specify the graph H whenever the context is clear. The notions not defined here are found in [2, 13].

2. k-Distance Secure Sets

Let $G = (V,E)$ be a graph and k be a positive integer less than or equal to diam(G). Throughout the discussion, we consider S to be a non empty subset of V. The subgraph induced by S is denoted by $\langle S\rangle$.

Definition 2.1. [9] *A k-distance attack on $S \subseteq V$ is a map $A : N_k[S] - S \to Bord(S)$ such that $d_H(x, A(x)) \leq k \ \forall \ x \in N_k[S] - S$, where $H = \langle (V-S) \cup \{A(x)\}\rangle$. A k-distance defense of S is a map $D : N_k[Bord(S)] \cap S \to Bord(S)$ such that $d_{\langle S\rangle}\Big(x, D(x)\Big) \leq k \ \forall \ x \in N_k[Bord(S)] \cap S$.*

Remark 2.2. *If A is a k-distance attack on S, then for any $x \in N_k[S] - S$, there must exist a path P between x and $A(x)$ with following properties.*

(1) The length of P is at most k.
(2) All the vertices other than $A(x)$ belong to $V - S$.

Any path with above properties is called an attacking path of x to $A(x)$ with respect to A.

Remark 2.3. *If D is a k-distance defense of S, then for any $x \in S$, there exists a path between x and $D(x)$ of length at most k in $\langle S\rangle$. Any such path is called a defending path of $D(x)$ by x with respect to D.*

Definition 2.4. [9] *A k-distance attack A on S is defendable if there is a k-distance defense D such that $|D^{-1}(z) \cap N_i^{\langle S\rangle}[z]| \geq |A^{-1}(z) \cap N_i^{\langle (V-S)\cup\{z\}\rangle}[z]|$ $\forall \ i \in \{1, 2, \ldots, k\}$ and $z \in Bord(S)$. Further the defense D is called a successful defense of S against the attack A and we say that D defends S against A.*

Definition 2.5. [9] *A non empty set $S \subseteq V$ is a k-distance secure set if every k-distance attack on S is defendable.*

For any graph $G = (V,E)$ and $X \subseteq V$, G_X denotes the subgraph of G with vertex set V and edge set $E - E(\langle X\rangle)$.

Theorem 2.6. [9] *A non empty set S is a k-distance secure set if and only if for any component $\langle C\rangle$ of $\langle S\rangle$ and for any $X \subseteq Bord(C)$,*

$$|N_i^{\langle C\rangle}[X]| \geq |N_i^{\langle (V-S)\cup C\rangle_C}[X] - S| \quad \forall\, i, 1 \leq i \leq k.$$

3. k-Distance Ultra Secure Sets

A special type of secure sets is introduced and studied in [10, 11]. A set $S \subseteq V$ is ultra secure[10] if there is a defense of S, which defends S against every attack on it. This can be extended to k-distance secure sets as follows.

Definition 3.1. *A non empty set $S \subseteq V$ is said to be a k-distance ultra secure set if there exists a k-distance defense of S which defends S against every k-distance attack on S.*

Let $S = \{i_1, i_2, \ldots, i_m, b_1, b_2, \ldots, b_s\} \subseteq V$ and $Bord(S) = \{b_1, b_2, \ldots, b_s\}$. To derive a characterization of a k-distance ultra secure set, we construct a new graph $G_k(S)$ by applying following steps.

Step-1: Let $i = 1$. Initialize $H := \langle S\rangle$ and $x := b_1$. Go to Step-2.

Step-2: Let $H(x)$ be the graph obtained from $\langle N_k^{\langle (V-S)\cup\{x\}\rangle}[x]\rangle$ by relabeling the vertices other than x as $y_{x1}, y_{x2}, \ldots, y_{xt_x}$. Let $H' = H \cup H(x)$. Go to Step-3.

Step-3: If $i < s$, then reset $i := i+1$, $H := H'$, $x := b_i$ and go to Step-2. If $i = s$, then stop the process.

An example for the construction of $G_k(S)$ is given in Figure 1.

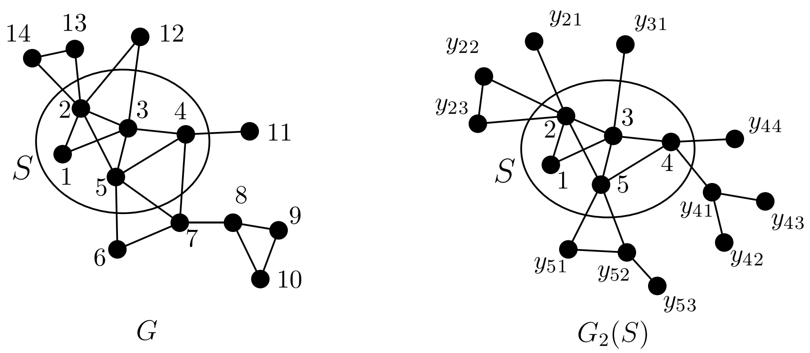

Figure 1. Construction of $G_2(S)$ from graph G with $S = \{1, 2, 3, 4, 5\}$.

By the construction, $\langle S\rangle$ is a subgraph of $G_k(S)$ and the border of S in $G_k(S)$ is same as that in G. Thus we may use the same notation for the border of S in both G as well as $G_k(S)$.

Lemma 3.2. *Let $G = (V, E)$ be a graph. For a non empty $S \subseteq V$, there exists only one k-distance attack A_S on S in $G_k(S)$. Further, S is a k-distance ultra secure set in G if and only if A_S is defendable in $G_k(S)$.*

Proof. Let A be a k-distance attack on S in $G_k(S)$. For any $z \in Bord(S)$, we denote $\langle (V(G_k(S)) - S) \cup \{z\}\rangle$ by H_z. By the construction of $G_k(S)$, it can be observed that for any distinct $x, y \in Bord(S)$, $(N_k^{H_x}[x] - S) \bigcap (N_k^{H_y}[y] - S) =$

$\emptyset$. Hence $A^{-1}(x) = N_k^{H_x}(x)$ for every $x \in Bord(S)$. Therefore there exists only one k-distance attack on S in $G_k(S)$. Denote this attack by A_S.

Suppose S is a k-distance ultra secure set in G. Then there exists a defense D of S such that for any attack A on S, $|D^{-1}(z) \cap N_i^{\langle S\rangle}[z]| \geq |A^{-1}(z) \cap N_i^{\langle (V-S)\cup\{z\}\rangle}[z]|\ \forall\ i,\ 1 \leq i \leq k$, for every $z \in Bord(S)$. In particular this holds for any attack A_z satisfying $A_z^{-1}(z) \cap N_i[z] = N_i^{\langle (V-S)\cup\{z\}\rangle}[z] - S\ \forall\ i$, $1 \leq i \leq k$. Note that for any $z \in Bord(S)$ and for any i with $1 \leq i \leq k$, $|A_S^{-1}[z] \cap N_i^{H_z}[z]| = |N_i^{\langle (V-S)\cup\{z\}\rangle}[z] - S| \leq |D^{-1}(z) \cap N_i^{\langle S\rangle}[z]|$. As D is a defense of S in $G_k(S)$, A_S is defendable in $G_k(S)$.

Conversely, assume that A_S is defendable in $G_k(S)$. Then there exists a defense D of S in $G_k(S)$ such that $|D^{-1}(z) \cap N_i^{\langle S\rangle}[z]| \geq |A_S^{-1}[z] \cap N_i^{H_z}[z]|$ for any i with $1 \leq i \leq k$ and for any $z \in Bord(S)$. Let A be any attack on S in G. Then for any $z \in Bord(S)$, $|A^{-1}(z) \cap N_i^{\langle (V-S)\cup\{z\}\rangle}[z]| \leq |N_i^{\langle (V-S)\cup\{z\}\rangle}[z] - S| = |N_i^{H_z}(z)| = |A_S^{-1}[z] \cap N_i^{H_z}[z]| \leq |D^{-1}(z) \cap N_i^{\langle S\rangle}[z]|\ \forall$ i, $1 \leq i \leq k$. Since D is a defense of S in G, it follows that S is a k-distance ultra secure set in G.

□

Lemma 3.3. *For a non empty $S \subseteq V$, a k-distance attack A_S on S is defendable in $G_k(S)$ if and only if for every component $\langle C\rangle$ of $\langle S\rangle$ and every $X \subseteq Bord(C)$,*

$$|N_i^{\langle C\rangle}[X]| \geq \sum_{x\in X} |N_i^{\langle (V-S)\cup\{x\}\rangle}[z] - S| \quad \forall\ i, 1 \leq i \leq k.$$

Proof. Let a k-distance attack A_S on S be defendable in $G_k(S)$. Then there exists a defense D of S in $G_k(S)$ such that for any $z \in Bord(S)$, $|D^{-1}(z) \cap N_i^{\langle S\rangle}[z]| \geq |A_S^{-1}(z) \cap N_i^{H_z}[z]|\ \forall\ i,\ 1 \leq i \leq k$. Let $\langle C\rangle$ be any component of $\langle S\rangle$ and $X \subseteq Bord(C)$ be any set. Then for any i with $1 \leq i \leq k$,

$$\begin{aligned} |N_i^{\langle C\rangle}[X]| \geq |\bigcup_{z\in X} \left(D^{-1}(z) \cap N_i^{\langle S\rangle}[z]\right)| &\geq |\bigcup_{z\in X} \left(A_S^{-1}(z) \cap N_i^{H_z}[z]\right)| \\ &= \sum_{z\in X} |A_S^{-1}(z) \cap N_i^{H_z}[z]| \\ &= \sum_{z\in X} |N_i^{H_z}[z] - S| \\ &= \sum_{z\in X} |N_i^{\langle (V-S)\cup\{z\}\rangle}[z] - S|. \end{aligned}$$

Conversely, assume that the above condition holds. Since $(N_k^{H_x}[x] - S) \cap (N_k^{H_x}[x] - S) = \emptyset$ for any distinct $x, y \in Bord(S)$, $|N_i^{\langle (V(G_k(S))-S)\cup C\rangle_C}[X] - S| = \sum_{x\in X} |N_i^{H_x}[x] - S|\ \forall\ i$ with $1 \leq i \leq k$, for any component $\langle C\rangle$ of $\langle S\rangle$ and $X \subseteq Bord(C)$. Then by Theorem 2.6, S is a k-distance secure set in $G_k(S)$. Equivalently A_S is defendable in $G_k(S)$. □

Lemma 3.2 and Lemma 3.3 together provide the following characterization of k-distance ultra-secure sets.

Theorem 3.4. *A non empty set $S \subseteq V$ is a k-distance ultra secure set if and only if for every component $\langle C\rangle$ of $\langle S\rangle$ and for any $X \subseteq Bord(C)$,*

$$|N_i^{\langle C\rangle}[X]| \geq \sum_{x\in X} |N_i^{\langle (V-S)\cup\{x\}\rangle}[x] - S| \quad \forall\, i, 1 \leq i \leq k.$$

4. Generalized k-Distance Secure Sets

Remark 2.2 and Remark 2.3 indicate that certain conditions are imposed on attacking and defending paths while defining a k-distance attack and a k-distance defense. However, one might consider generalizing secure sets even by removing these conditions on attacking and defending paths. This gives rise to the following types of attacks and defenses.

Definition 4.1. *A general k-distance attack on S is a function $A : N_k[S] - S \to S$ such that $d(x, A(x)) \leq k\ \forall\ x \in N_k[S] - S$. A general k-distance defense of S is function $D : S \to S$ such that $d(x, D(x)) \leq k\ \forall\ x \in S$. A pure k-distance attack on S is a function $A : N_k[S] - S \to S$ such that $d_H(x, A(x)) \leq k\ \forall\ x \in N_k[S] - S$, where $H = \langle (V - S) \cup \{A(x)\}\rangle$. A pure k-distance defense of S is a function $D : S \to S$ such that $d_{\langle S\rangle}\Big(x, D(x)\Big) \leq k$ $\forall\ x \in S$.*

Remark 4.2. *A pure k-distance attack coincides with a k-distance attack given in Definition 2.1. Similarly, a pure k-distance defense corresponds to a k-distance defense given in Definition 2.1 when the domain and co-domain are restricted to $N_k[Bord(S)] \cap S$ and $Bord(S)$ respectively.*

A general k-distance attack A on S is defendable by a general k-distance defense D of S if for every $z \in S$, $|D^{-1}(z) \cap N_i[z]| \geq |A^{-1}(z) \cap N_i[z]|$ $\forall\ i,\ 1 \leq i \leq k$. A general k-distance attack A of S is defendable by a pure k-distance defense D of S if for every $z \in S$, $|D^{-1}(z) \cap N_i^{\langle S\rangle}[z]| \geq |A^{-1}(z) \cap N_i[z]|\ \forall\, i, 1 \leq i \leq k$. A pure k-distance attack A of S is defendable by a general k-distance defense D of S if for every $z \in S$, $|D^{-1}(z) \cap N_i[z]| \geq |A^{-1}(z) \cap N_i^H[z]|\ \forall\ i,\ 1 \leq i \leq k$, where $H = \langle (V - S) \cup \{z\}\rangle$. A pure k-distance attack A of S is defendable by a pure k-distance defense D of S if for every $z \in S$, $|D^{-1}(z) \cap N_i^{\langle S\rangle}[z]| \geq |A^{-1}(z) \cap N_i^H[z]|\ \forall\ i,\ 1 \leq i \leq k$, where $H = \langle (V - S) \cup \{z\}\rangle$.

Definition 4.3. *A non empty set $S \subseteq V$ is a*

(1) k-distance (G, G) secure set if every general k-distance attack on S is defendable by a general k-distance defense.
(2) k-distance (G, P) secure set if every general k-distance attack on S is defendable by a pure k-distance defense.
(3) k-distance (P, G) secure set if every pure k-distance attack on S is defendable by a general k-distance defense.
(4) k-distance (P, P) secure set if every pure k-distance attack on S is defendable by a pure k-distance defense.

Remark 4.4. *Let G be any graph. Then*

(1) all the above types of secure sets coincide with secure sets defined in [1] *when $k = 1$, .*

(2) a k-distance (P, P) secure set coincides with a k-distance secure sets defined in [9].

By the definition, it is clear that a pure k-distance attack/defense is also a general k-distance attack/defense. This leads to the following.

Remark 4.5. *In any graph G, let $S \subseteq V$. Then*

(1) S is a k-distance (G, P) secure set $\implies$ S is a k-distance (G, G) secure set $\implies$ S is a k-distance (P, G) secure set.

(2) S is a k-distance (G, P) secure set $\implies$ S is a k-distance (P, P) secure set $\implies$ S is a k-distance (P, G) secure set.

Theorem 4.6. *Every k-distance (P, P) secure set is a k-distance (G, G) secure set.*

Proof. Let S be a k-distance (P, P) secure set and A be any general k-distance attack on S. From attack A, a pure k-distance attack A' on S can be constructed as follows. Let $x \in N_k[S] - S$ be arbitrary. Consider a shortest path P_x between x and $A(x)$ in G. Clearly $P_x \cap Bord(S) \neq \emptyset$. Let $t_x = \min\{d(x, z) : z \in P_x \cap Bord(S)\}$. Let $y_x \in P_x \cap Bord(S)$ such that $d(x, y_x) = t_x$. Then define $A'(x) = y_x$. Since S is a k-distance (P, P) secure set, there exists a pure k-distance defense D which defends S against A'. From D, a general k-distance defense D' of S can be constructed as follows.

Since D defends S against A', for every $x \in N_k[S] - S$, there exists $z_x \in N_k[Bord(S)] \cap S$ such that $d_{\langle S\rangle}(y_x, z_x) \leq d_{\langle (V-S)\cup\{y_x\}\rangle}(y_x, x) = d(x, y_x)$. Thus $d(y_x, z_x) \leq d_{\langle S\rangle}(y_x, z_x) \leq d_{\langle (V-S)\cup\{y_x\}\rangle}(y_x, x)$. Note that $d(x, A(x)) = d(x, y_x) + d(y_x, A(x)) \geq d(y_x, z_x) + d(y_x, A(x)) \geq d(z_x, A(x))$. Now, define $D' : S \to S$ such that for any $z \in S$, $D'(z) = A(x)$ whenever $z = z_x$ for some $x \in N_k[S] - S$ and $D'(z) = z$ otherwise. Then D' is a general k-distance defense of S which defends S against A. Since A is arbitrary, S is a k-distance (G, G) secure set. □

Remark 4.7. *A k-distance (G, G) secure set need not be a k-distnace (P, P) secure set. In Figure 2, the set $S = \{1, 2, 5\}$ is a 2-distance (G, G) secure set, which is not a 2-distance (P, P) secure set.*

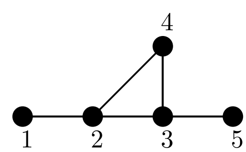

FIGURE 2

Theorem 2.6 has already characterized k-distance (P, P) secure sets. The following classical theorem due to P. Hall is useful to derive a characterization of a k-distance (G, P) secure set.

Theorem 4.8. [5, 6, 12] *Suppose $S_1, S_2, \ldots, S_n$ are sets and $k_1, k_2, \ldots, k_n$ are non negative integers. There exist mutually disjoint sets $D_1, D_2, \ldots, D_n$ such that $D_i \subseteq S_i$, $|D_i| = k_i$ for $1 \leq i \leq n$ if and only if $|\bigcup_{i\in I} S_i| \geq \sum_{i\in I} k_i$ for any $I \subseteq \{1, 2, \ldots, n\}$.*

The following theorem characterizes k-distance (G, P) secure sets.

Theorem 4.9. *A non empty set $S \subseteq V$ is a k-distance (G, P) secure set if and only if for every $X \subseteq S$, $|N_i^{\langle S \rangle}[X]| \geq |N_i[X] - S|$ $\forall$ i, $1 \leq i \leq k$.*

Proof. Let S be k-distance (G, P) secure set and A be a general k-distance attack on S. Then there exists a pure k-distance defense of S such that for any $z \in S$, $|D^{-1}(z) \cap N_i^{\langle S \rangle}[z]| \geq |A^{-1}(z) \cap N_i[z]|$ $\forall$ i, $1 \leq i \leq k$. Let $X \subseteq S$ be arbitrary. Then for any i, $1 \leq i \leq k$,

$$\begin{aligned} |N_i^{\langle S \rangle}[X]| = |\bigcup_{z \in X} N_i^{\langle S \rangle}[z]| &\geq |\bigcup_{z \in X} (D^{-1}(z) \cap N_i^{\langle S \rangle}[z])| \\ &= \sum_{z \in X} |D^{-1}(z) \cap N_i^{\langle S \rangle}[z]| \\ &\geq \sum_{z \in X} |A^{-1}(z) \cap N_i[z]| \\ &= |\bigcup_{z \in X} (A^{-1}(z) \cap N_i[z])|. \end{aligned}$$

Note that there is a general k-distance attack A on S with $\bigcup_{z \in X} (A^{-1}(z) \cap N_i[z]) = N_i[X] - S$ $\forall$ i, $1 \leq i \leq k$, which can be obtained by mapping every vertex of $N_i[X] - S$ to its nearest vertex in X. Then by the above argument, it clearly follows that $|N_i^{\langle S \rangle}[X]| \geq |N_i[X] - S|$ $\forall$ i, $1 \leq i \leq k$.

Conversely, suppose that for every $X \subseteq S$, $|N_i^{\langle S \rangle}[X]| \geq |N_i[X] - S|$ $\forall$ i, $1 \leq i \leq k$. Let A be any general k-distance attack on S. Then for any $z \in S$ and for any i, $1 \leq i \leq k$, let $P_{iz} = N_i^{\langle S \rangle}[z]$ and $k_{iz} = |A^{-1}(z) \cap \big(N_i[z] - N_{i-1}[z]\big)|$. Let $X \subseteq S$ and $I \subseteq \{1, 2, \ldots, k\}$ be arbitrary. Let t be the maximum of I. Then

$$\begin{aligned} |\bigcup_{i \in I, z \in X} P_{iz}| = |N_t^{\langle S \rangle}[X]| &\geq |N_t[X] - S| \\ &\geq |\bigcup_{z \in X} \big(A^{-1}(z) \cap N_t[z]\big)| \\ &\geq \sum_{z \in X} \sum_{i=1}^{t} k_{iz} \\ &\geq \sum_{z \in X, i \in I} k_{iz}. \end{aligned}$$

Thus, by Theorem 4.8, for every i with $1 \leq i \leq k$ and for all $z \in S$, there exist mutually disjoint sets $D_{iz} \subseteq P_{iz}$, with cardinality k_{iz}. Then define a map $D : S \to S$ by

$$D(x) = \begin{cases} z & \text{if } x \in D_{iz} \\ x & \text{otherwise} \end{cases} \quad \forall \ x \in S.$$

Then D is a pure k-distance defense of S with $|D^{-1}(z) \cap N_i^{\langle S \rangle}[z]| \geq |A^{-1}(z) \cap N_i[z]|$ $\forall\, i, 1 \leq i \leq k$ and $z \in S$. Thus, S is a k-distance (G, P) secure set. $\square$

The following results are similar to Theorem 4.9.

Theorem 4.10. *A non empty set $S \subseteq V$ is a k-distance (G,G) secure set if and only if for every $X \subseteq S$, $|N_i[X] \cap S| \geq |N_i[X] - S|$ $\forall$ i, $1 \leq i \leq k$.*

Theorem 4.11. *A non empty set $S \subseteq V$ is a k-distance (P,G) secure set if and only if for every $X \subseteq Bord(S)$, $|N_i[X] \cap S| \geq |N_i^{\langle (V-S)\cup Bord(S)\rangle_{Bord(S)}}[X] - S|$ $\forall$ i, $1 \leq i \leq k$.*

5. Conclusion

In this paper, generalized k-distance secure sets are introduced and their interrelationships are studied. The minimum cardinality of a k-distance secure set in a graph G is called k-distance security number of G, denoted by $ds_k(G)$. Similarly one may define the generalized distance security numbers and study their properties.

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